

# The bootstrap condition for many reggeized gluons and the photon structure function at low $x$ and

$$N_c \rightarrow \infty$$

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## Abstract.

The bootstrap condition is generalized to  $n$  reggeized gluons. As a result it is demonstrated that the intercept generated by  $n$  reggeized gluons cannot be lower than the one for  $n = 2$ . Arguments are presented that in the limit  $N_c \rightarrow \infty$  the bootstrap condition reduces the  $n$  gluon chain with interacting neighbours to a single BFKL pomeron. In this limit the leading contribution from  $n$  gluons corresponds to  $n/2$  non-interacting BFKL pomerons (the  $n/2$  pomeron cut). The sum over  $n$  leads to a unitary  $\gamma^* \gamma$  amplitude of the eikonal form.

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## 1 Introduction.

Recently much attention has been given to the system of many reggeized gluons with a pair interaction between them in the framework of the equation proposed by J.Bartels, J. Kwiecinski and M.Praszalowicz [ 1,2 ]. Mostly the case of infinite number of colours,  $N_c \rightarrow \infty$ , was investigated, when the colour structure of the equation drastically simplifies. The resulting chain Hamiltonian, with only the neighbours interacting, has been extensively studied in [ 3-6 ]. The conformal symmetry seems to reduce the problem to a completely integrable one but it still remains very complicated.

A close study of the old results on the two-gluon case and the results by J.Bartels on the three-and four-gluon cases [ 7 ] suggests, however, that the physical solutions depend heavily on the form of the inhomogeneous term in the equation, that is, the gluon coupling to the target (projectile). It may occur that although the complete spectrum of the  $n$ -gluon Hamiltonian is complicated and hardly calculable, the physical solutions corresponding to a given coupling to the external source are simple and easy to find. It is this aspect of the  $n$ -gluon problem, which is studied in the present paper. We do not touch here the problem of the introduction of the running coupling constant into the formalism. The coupling constant is assumed to be fixed and small throughout the paper.

To have a well-defined external gluon source, we restrict ourselves to the interaction of photons, real or virtual. A photon splits into a  $q\bar{q}$  pair which emits gluons in a well-defined and calculable manner. In the lowest order in the coupling constant it corresponds to the coupling of  $n$  gluons to a single  $q\bar{q}$  loop. For this particular target (projectile) we argue that in the limit  $N_c \rightarrow \infty$  the physical solutions in the  $n$ -gluon system are quite simple. They separate into two classes depending on the colour structure. One class of solutions corresponds to the gluons pairing into colour singlets. Then the interaction between different pairs vanishes in the limit  $N_c \rightarrow \infty$  and the solution reduces to the product of  $n/2$  independent BFKL pomerons [ 8 ], i.e. to the standard  $n/2$  pomeron cut. The other solution corresponds to the gluons being in the vector (adjoint) colour representation with their neighbours. It is for this colour configuration that the Hamiltonian acquires the chain form studied in [ 3-6 ]. Our result is that in this case, due to the so-called bootstrap condition discovered for two gluons in [ 9 ] and generalized for  $n$  gluons here, all gluons coalesce into a single pair, that is, the solution for  $n$  gluons reduces to a single BFKL pomeron. To clarify the meaning of this result, we stress that by no means do we assert that all eigenstates of the corresponding Hamiltonian have this structure. Our result is that only a particular class of solutions, which are precisely the ones of physical interest, reduce to a single pomeron.

We have also to stress that our results in this respect rest heavily on those by J.Bartels,

who studied the system of four reggeized gluons with  $N_c = 3$  in [ 7 ]. Our investigation was, in fact, inspired by this paper. We also mention the results by G.Korchemsky [ 6 ] who using a sophisticated one-dimensional lattice technique conjectured that the minimal energy of the  $n$  gluon lattice coincides with the one for  $n = 2$ .

## 2 The bootstrap condition for arbitrary number of gluons

We start by recalling the bootstrap condition for two reggeized gluons as obtained in [ 9 ]. The two-gluon equation can be written as an inhomogeneous Schroedinger equation

$$(H - E)\psi = F \quad (1)$$

with the Hamiltonian for the colour group  $SU(N_c)$

$$H = (N_c/2)(t(q_1) + t(q_2)) + (T_1 T_2)V \quad (2)$$

Here  $(N_c/2)t(q)$  is the kinetic energy of the gluon given by its Regge trajectory

$$t(q) = g^2 \int (d^2 q_1 / (2\pi)^3) \frac{q^2}{q_1^2 q_2^2}, \quad q_1 + q_2 = q \quad (3)$$

Infrared regularization is understood here and in the following where necessary (either dimensional or by a nonzero gluon mass). The interaction term, apart from the product of the gluon colour vectors  $T_i^a$ ,  $i = 1, 2$ ,  $a = 1, \dots, N_c$ , involves the BFKL kernel [ 8 ]

$$(1/g^2)V(q_1, q_2, q'_1, q'_2) = \left( \frac{q_1^2}{q'_1{}^2} + \frac{q_2^2}{q'_2{}^2} \right) \frac{1}{(q_1 - q'_1)^2} - \frac{q^2}{q'_1{}^2 q'_2{}^2} \quad (4)$$

with  $q_1 + q_2 = q'_1 + q'_2$ . Comparing (3) with (4) one finds the bootstrap condition

$$\int (d^2 q_1 / (2\pi)^3) V(q_1, q_2, q'_1, q'_2) = t(q_1) + t(q_2) - t(q_1 + q_2) \quad (5)$$

It is customary to discuss a homogeneous equation (1) (with  $F = 0$ ) to seek for the ground state energy, which determines the rightmost singularity in the complex angular momentum  $j$  related to the energy by

$$j = 1 - E \quad (6)$$

For our purpose it is, however, essential to conserve the inhomogeneous term  $F(q_1, q_2)$ , which represents the two-gluon-particle vertex.

Consider the vector colour channel with  $T_1 T_2 = -N_c/2$  and, most important, assume that the inhomogeneous term depends only on the total momentum of the two gluons:  $F = F(q_1 + q_2)$ . Then using the bootstrap relation (5) one easily finds the solution to Eq. (1):

$$\psi(q_1, q_2) = \psi(q_1 + q_2) = F(q_1 + q_2) / (t(q_1 + q_2) - E) \quad (7)$$

It means that the two gluons 1 and 2 have fused into a single one with the momentum  $q_1 + q_2$ .

We have to stress two points important for the following. First, the solution (7) is the correct and unique solution for the given form of the inhomogeneous term  $F = F(q_1 + q_2)$ . It does not involve all possible eigenstates of the Hamiltonian, which may be quite complicated and which might appear should the inhomogeneous term depend on  $q_1$  and  $q_2$  separately. Still if we know that the external coupling has a specific dependence  $F(q_1 + q_2)$  the found solution is the desired one and we need not bother about other eigenstates of the Hamiltonian. Second, the solution refers to both signatures, positive and negative. The solution with the negative signature physically represents the reggeized gluon itself. It is this solution which is meant by the bootstrap. However the same singularity in  $j$  appears also in the positive signature where it does not correspond to any particle at  $j = 1$ . We thus observe degeneration in signature.

We pass now to the main topic of this section: a generalization of this result to the  $n$ -gluon case. For  $n$  gluons Eq. (1) holds with the Hamiltonian which is a sum of kinetic energies and pair interactions:

$$H = (N_c/2) \sum_{i=1}^n t(q_i) + \sum_{i < k}^n (T_i T_k) V_{ik} \quad (8)$$

Here  $V_{ik}$  is the interaction of gluons  $i$  and  $k$  with the kernel  $V(q_i, q_k, q'_i, q'_k)$ . We do not impose any restrictions on the total colour  $T = \sum_{i=1}^n T_i$ . (However only for  $T = 0$  the Hamiltonian is infrared stable).

Assume now that  $T_1 T_2 = -N_c/2$ , i.e. the gluons 1 and 2 form a colour vector. Then we claim that for a certain specific choice of the inhomogeneous term  $F$  Eq. (1) for  $n$  gluons reduces to that for  $n - 1$  gluons, the gluons 1 and 2 fused into a single gluon which carries their total momentum and colour. In other words, as in the two-gluon case, a pair of gluons in the adjoint representation is equivalent to a single gluon.

Of course, a specific form of the inhomogeneous term is a decisive instrument for bootstrapping the two gluons 1 and 2. For  $n$  gluons we choose

$$F_n(q_1, q_2, \dots, q_n) = \sum_{i=3}^n \int (d^2 q'_i / (2\pi)^3) \hat{W}(q_1, q_2, q_i; q'_1 q'_i) \psi(q'_1, q_3, \dots, q'_i, \dots, q_n) + F_{n-1}(q_1 + q_2, q_3, \dots, q_n) \quad (9)$$

with  $q'_1 + q'_i = q_1 + q_2 + q_i$ . In Eq. (9)  $\psi_{n-1}$  is the solution of the Schrödinger equation (1) with the inhomogeneous term  $F_{n-1}$ . Gluon 1 in it substitutes the fused initial gluons 1 and 2. The kernel  $\hat{W}$  is also an operator acting on colour indeces of  $\psi_{n-1}$ . It is convenient to retain a pair of colour indeces for gluon 1 in  $\psi_{n-1}$  inherited from the initial gluons 1 and 2 by means of a projector onto the adjoint representation in the colour  $T_1 + T_2$ . This allows to consider  $\hat{W}$  as an operator acting in the colour space of  $n$  gluons. Then it has the form

$$\hat{W}(q_1, q_2, q_i; q'_1, q'_i) = (T_1 T_i) W(q_1, q_2, q_i; q'_1, q'_i) + (T_2 T_i) W(q_2, q_1, q_i; q'_1, q'_i) \quad (10)$$

the momentum space kernel  $W$  being a difference between two BFKL kernels

$$(1/g)W(q_1, q_2, q_i; q'_1, q'_i) = V(q_1 + q_2, q_i, q'_1, q'_i) - V(q_1, q_i, q'_1 - q_2, q'_i) \quad (11)$$

Let us demonstrate that with the inhomogeneous term  $F_n$  given by Eq. (9) and  $T_1 T_2 = -N_c/2$  the Schrödinger equation (1) is solved by

$$\psi_n(q_1, q_2, q_3, \dots, q_n) = g\psi_{n-1}(q_1 + q_2, q_3, \dots, q_n) \quad (12)$$

Indeed putting (12) into the equation we find that the interaction term  $(T_1 T_2)V_{12}$ , according to (5), substitutes the sum of kinetic terms  $t(q_1) + t(q_2)$  for the gluons 1 and 2 by  $t(q_1 + q_2)$  which is precisely the kinetic term for the function  $\psi_{n-1}(q_1 + q_2, q_3, \dots)$ . The interaction of gluon 1 with the  $i$ th one,  $i \geq 3$ , takes the form

$$\begin{aligned} (T_1 T_i) \int (d^2 q'_i / (2\pi)^3) V(q_1, q_i, q'_1, q'_i) g\psi_{n-1}(q'_1 + q_2, q_3, \dots, q'_i, \dots, q_n) = \\ (T_1 T_i) \int (d^2 q'_i / (2\pi)^3) V(q_1, q_i, q'_1 - q_2, q'_i) g\psi_{n-1}(q'_1, q_3, \dots, q'_i, \dots, q_n) \end{aligned} \quad (13)$$

The momentum is conserved during the interaction, so that on the lefthand side  $q_1 + q_i = q'_1 + q'_i$  and on the righthand side  $q_1 + q_2 + q_i = q'_1 + q'_i$ . The term (13) is cancelled by the identical contribution with the opposite sign coming from the inhomogeneous term  $F_n$  (the second term in (11) in the part of  $\hat{W}$  proportional to  $T_1 T_i$ ). Instead of it from the inhomogeneous term comes the contribution (the first term in (11))

$$(T_1 T_i) \int (d^2 q'_i / (2\pi)^3) V(q_1 + q_2, q_i, q'_1, q'_i) g\psi_{n-1}(q'_1, q_3, \dots, q'_i, \dots, q_n) \quad (14)$$

In the same manner the interaction of gluon 2 with the  $i$ th one

$$\begin{aligned} (T_2 T_i) \int (d^2 q'_i / (2\pi)^3) V(q_2, q_i, q'_2, q'_i) g\psi_{n-1}(q_1 + q'_2, q_3, \dots, q'_i, \dots, q_n) = \\ (T_2 T_i) \int (d^2 q'_i / (2\pi)^3) V(q_2, q_i, q'_2 - q_1, q'_i) g\psi_{n-1}(q'_2, q_3, \dots, q'_i, \dots, q_n) \end{aligned} \quad (15)$$

is transformed by the term in  $F_n$  proportional to  $T_2 T_i$  into

$$(T_2 T_i) \int (d^2 q'_i / (2\pi)^3) V(q_1 + q_2, q_i, q'_1, q'_i) g\psi_{n-1}(q'_1, q_3, \dots, q'_i, \dots, q_n) \quad (16)$$

The two terms (14) and (16) sum into

$$(T_1 + T_2, T_i) \int (d^2 q'_i / (2\pi)^3) V(q_1 + q_2, q_i, q'_1, q'_i) g\psi_{n-1}(q'_1, q_3, \dots, q'_i, \dots, q_n) \quad (17)$$

which is precisely the correct form for the interaction of the gluon with the total colour  $T_1 + T_2$  and momentum  $q_1 + q_2$  in which the initial gluons 1 and 2 have fused. Other kinetic energies  $t(q_i)$  and interactions  $(T_i T_k)V_{ik}$  with  $i, k \geq 3$  are not influenced by the bootstrap of gluons 1

and 2. As a result we obtain the Schroedinger equation (1) for the function  $\psi_{n-1}$  describing  $n - 1$  gluons with the inhomogeneous term  $F_{n-1}$ .

The obtained result immediately allows to conclude that the leading singularity in the complex momentum  $j$  for the system of  $n$  gluons (the ground state energy) cannot in any case be lower (higher) than for the system of  $n - 1$  gluons. In fact, for any solution for  $n - 1$  gluons we can construct a solution for  $n$  gluons using  $F_n$  given by (8) as the inhomogeneous term. Note that  $F_n$  depends on energy  $E$  via the solution  $\psi_{n-1}$  and thus possesses all branch points in  $E$  which mark thresholds in the spectrum for  $n - 1$  gluons. The fact that the final solution  $\psi_n$  for  $n$  gluons, according to (12), has all these branch points and no other ones means that the energy thresholds in the spectrum for  $n$  gluons include all those for  $n - 1$  gluons (and maybe some new ones). This conclusion is in line with the conjecture by G.Korchemsky [ 6 ], based on the Yang-Baxter equation formalism applied to the gluon chain Hamiltonian in the limit  $N_c \rightarrow \infty$ . Note, however, that our result is much stronger: it applies to finite  $N_c$  and the full gluon Hamiltonian (8).

To conclude this section we present the explicit form of the kernel  $W$  (for  $i = 3$ )

$$(1/g)W(q_1, q_2, q_3; q'_1, q'_3) = \frac{q_{12}^2}{q'^2_1(q_3 - q'_3)^2} + \frac{q_{13}^2}{q'^2_3(q_2 - q'_1)^2} - \frac{q_{123}^2}{q'^2_1 q'^2_3} - \frac{q_1^2}{(q'_1 - q_2)^2(q_3 - q'_3)^2} \quad (18)$$

where  $q_{12} = q_1 + q_2$  etc. This kernel (multiplied by  $q'^2_1 q'^2_3$  and with the opposite sign) was obtained by J.Bartels as a vertex for transition from two to three reggeized gluons in the four-gluon system, by studying the three-particle amplitude in the triple Regge kinematics [ 7 ]. Our result may be considered as a generalization to an arbitrary number of gluons in an arbitrary colour state.

The described bootstrap of gluons takes place for any number of colours. However it leads to spectacular results in the limit  $N_c \rightarrow \infty$ , which will be studied presently.

### 3 The colour structure for $N_c \rightarrow \infty$ . Multipomeron solutions

It is well-known that in the limit  $N_c \rightarrow \infty$  the colour group  $SU(N_c)$  does not differ from  $U(N_c)$  and the gluon may be represented by a  $q\bar{q}$  pair. Then the colour trace can be taken by following quark lines. The leading contribution is obtained from diagrams in which gluon lines do not cross. Taking as a target a  $q\bar{q}$  loop with external colourless sources (real or virtual photons) one obtains a cylinder built on this loop as a dominant configuration, each gluon interacting only with his two neighbours. In this cylinder each pair of neighbour gluons are in the adjoint representation. Indeed crossing two neighbouring gluon lines, each represented

by a  $q\bar{q}$  pair, one encounters the configuration

$$\sum_{\gamma=1}^{N_c} q_\alpha^{(1)} \bar{q}_\gamma^{(1)} q_\gamma^{(2)} \bar{q}_\beta^{(2)}, \quad \alpha, \beta = 1, \dots, N_c$$

which evidently transforms as a vector (it is a superposition of antisymmetric and symmetric representations with equal weights). This means that for the resulting chain Hamiltonian, each gluon interacting only with his two neighbours, all  $T_i T_k = -N_c/2$ . As a result one finds a colour independent chain Hamiltonian studied in [ 3-6 ].

Of course, the described colour structure, each pair of neighbours being in the adjoint representation, is not the only one possible at  $N_c \rightarrow \infty$ . The simplest configuration different from it is the one in which pairs of gluons form a colourless state. For the total colour zero it requires that the number of gluons  $n$  be even. Then the colour wave function takes the form

$$\chi_{a_1, \dots, a_n} = (N_c^2 - 1)^{-n/4} \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{a_{n-1} a_n} \quad (19)$$

each  $a_i$  taking the values from 1 to  $N_c^2 - 1$ . It is important that in such a configuration gluons belonging to different colourless pairs cannot couple to form a colourless pair or a vector. Indeed the probability to find, say, the gluons 1 and 3 in a colour representation  $R$  is determined by the projection  $P_R^{(13)} \chi$  onto this representation. For the colourless state  $R = (1)$  the projector is

$$(P_{(1)}^{(13)})_{a_1 a_3, a'_1 a'_3} = (N_c^2 - 1)^{-1} \delta_{a_1 a_3} \delta_{a'_1 a'_3} \quad (20)$$

Applying it to (19) we find

$$P_{(1)}^{(13)} \chi = (N_c^2 - 1)^{-n/4-1} \delta_{a_1 a_3} \delta_{a_2 a_4} \dots \delta_{a_{n-1} a_n} \quad (21)$$

with the norm

$$\|P_{(1)}^{(13)} \chi\|^2 = 1/(N_c^2 - 1)^2 \quad (22)$$

Thus in the limit  $N_c \rightarrow \infty$   $P_{(1)}^{(13)} \chi = 0$ , i.e. the gluons 1 and 2 are never in a colourless state. Likewise for the antisymmetric vector state

$$(P_{(N_c^2 - 1)_A}^{(13)})_{a_1 a_3, a'_1 a'_3} = c_A f_{a_1 a_3 a} f_{a'_1 a'_3 a} \quad (23)$$

where  $f$ 's are the structure constants of  $SU(N_c)$  and  $c_A$  is the appropriate normalization constant. We find

$$P_{(N_c^2 - 1)_A}^{(13)} \chi = c_A f_{a_1 a_3 a} f_{a'_2 a'_4 a} \delta_{a_5 a_6} \dots \delta_{a_{n-1} a_n} \quad (24)$$

and

$$\|P_{(N_c^2 - 1)_A}^{(13)} \chi\|^2 = 1/(N_c^2 - 1) \quad (25)$$

The same result holds for the symmetric vector representation  $(N_c^2 - 1)_S$  with  $f_{abc} \rightarrow d_{abc}$ . Thus although the probability to find the gluons 1 and 3 in a vector state is somewhat higher than for a colourless state, it still goes to zero as  $N_c \rightarrow \infty$ . In conclusion, in this limit

$$P_{(1)}^{(13)}\chi = P_{(N_c^2 - 1)_A}^{(13)}\chi = P_{(N_c^2 - 1)_S}^{(13)}\chi = 0 \quad (26)$$

Now recall that for the total colour of  $n$  gluons equal to zero the Hamiltonian (8) can be represented in the form [ 3 ]

$$H = -(1/2) \sum_{i < k}^n (T_i T_k) H_{ik} \quad (27)$$

where

$$H_{ik} = t(q_i) + t(q_k) - 2V_{ik} \quad (28)$$

is a pair Hamiltonian for gluons  $i$  and  $k$  acting only on space variables and infrared stable. For this pair of gluons in a given colour representation  $R$  we have

$$T_i T_k = (1/2) C_2(R) - N_c \quad (29)$$

where  $C_2$  is the quadratic Casimir operator. Two gluons can combine to form the following colour states: a colourless one (1), antisymmetric and symmetric vector states  $(N_c^2 - 1)_{A,S}$ , two other symmetric representations, which we denote generically as  $(S)$ , and two other antisymmetric representations  $(A)$ . The values of the operator  $T_i T_k$ , found from (29) for representations (1),  $(N^2 - 1)_A$ ,  $(N^2 - 1)_S$ ,  $(S)$ ,  $(A)$  are respectively

$$-N_c, -N_c/2, -N_c/2, \pm 1, 0 \quad (30)$$

From this we conclude that in the limit  $N_c \rightarrow \infty$  two gluons interact only when they are either in a colourless state or in a vector one. Other symmetric states give a contribution of a relative order  $1/N_c$  and other antisymmetric states decouple altogether.

Returning to our colorless pair configuration (19) we observe that gluons belonging to different pairs do not interact, since they cannot be in a colourless or vector state. Therefore the Hamiltonian for such a configuration reduces to a sum of  $n/2$  independent BFKL pair Hamiltonians (with  $T_{2i-1} T_{2i} = -N_c$ )

$$H = (N_c/2) \sum_{i=1}^{n/2} H_{2i-1, 2i} \quad (31)$$

The corresponding general solution of the homogeneous Schrödinger equation is a product of  $n/2$  BFKL pomerons

$$\psi_n(q_1, \dots, q_n) = \prod_{i=1}^{n/2} \psi_{BFKL}(q_{2i-1}, q_{2i}) \quad (32)$$

with the energy which is the sum of the pomeron energies. The ground state is evidently the BFKL pomeron one multiplied by  $n/2$ :

$$E_n = (n/2)E_0, \quad E_0 = -(g^2 N_c/\pi^2) \ln 2 \quad (33)$$

Physically this solution is nothing but a normal  $n/2$  cut. Our result is thus the cut is an exact solution of the Hamiltonian for  $n$  reggeized gluons in the large  $N_c$  limit.

In the following we shall argue that the  $n/2$  pomeron cut represents the leading singularity associated with  $n$  reggeized gluons in the limit  $N_c \rightarrow \infty$ . Its simple structure and a simple form for its interaction with a  $q\bar{q}$  pair make it possible to sum terms coming from arbitrary number of gluons and thus find the leading contribution, which satisfies the unitarity in the limit of large  $N_c$ . This will be done later, in Section 6., after we study the chain Hamiltonian contribution with neighbouring gluons in the vector colour state and show that this contribution is trivial.

Meanwhile we point out that the colourless pair configuration is not the only alternative to the pure vector configuration corresponding to a cylinder diagram. Evidently one may put the gluons in a mixed colour state. Some of them may couple into colourless pairs:  $(12), (34), \dots (2k-1, 2k)$ , the rest  $n - 2k$  gluons joining in a number of cylinder configurations. The Hamiltonian for such a state splits into a sum of  $k$  independent BFKL Hamiltonians for colourless pairs and chain Hamiltonians for the rest gluons

$$H = (N_c/2) \sum_{i=1}^k H_{2i-1, 2i} + (N_c/4) \sum_{i=2k+1}^n H_{i, i+1} \quad (34)$$

with last and first coordinates identical for each cylinder. Solutions to the corresponding Schrödinger equation will be given by a product of  $k$  BFKL pomerons and solutions to the chain Hamiltonian problem for the rest gluons.

According to [ 10 ], for a configuration of  $n$  gluons which splits into  $k$  colourless subconfigurations the colour factor is  $N_c^{n+2-2k}$ . For the dominant configuration, the  $n$ -gluon cylinder ( $k = 1$ ) the factor is  $N_c^n$  and for the colourless pair configuration ( $k = n/2$ ) it is  $N_c^2$ . Thus it may look as if the latter configurations can safely be neglected as  $N_c \rightarrow \infty$ . However, as we shall presently see, this is not true.

## 4 Solution to the gluon chain equation

The peculiarity of the gluon chain in the limit  $N_c \rightarrow \infty$  is that all gluons are in the vector colour state with respect to their neighbours and this property is conserved when one bootstraps a pair of them into a single gluon. This allows to apply the bootstrap mechanism

$(n - 2)$  times consecutively to fuse all  $n$  gluons into a pair in the colourless state, which is a BFKL pomeron.

For the gluon chain all  $T_i T_k = -N_c/2$  so that we need not bother about the colour variables. Since the gluons are interacting only with their neighbours, we have to retain only one of the two terms in (10) (except for  $n = 3$ ). The chain Hamiltonian acting only on the space variable is

$$H = \sum_{i=1}^n (t(q_i) - V_{i,i+1}), \quad n + 1 \equiv 1 \quad (35)$$

The inhomogeneous term for bootstrapping the gluons 1 and 2 can now be directly written in terms of the space kernel  $W$ :

$$\begin{aligned} F_n(q_1, \dots, q_n) &= - \int (d^2 q'_3 / (2\pi)^3) W(q_2, q_1, q_3; q'_1, q'_3) \psi_{n-1}(q'_1, q'_3, q_4, \dots, q_n) \\ &\quad - \int (d^2 q'_n / (2\pi)^3) W(q_1, q_2, q_n; q'_1, q'_n) \psi_{n-1}(q'_1, q_3, q_4, \dots, q'_n) + F_{n-1}(q_1 + q_2, q_3, \dots, q_n) \end{aligned} \quad (36)$$

With this form of  $F_n$  one finds that the solution to the Schroedinger equation with the Hamiltonian (35) for  $n$  gluons is given by

$$\psi_n(q_1, \dots, q_n) = g \psi_{n-1}(q_1 + q_2, q_3, \dots, q_n) \quad (37)$$

where  $\psi_{n-1}$  is the solution of the same equation for  $n - 1$  gluon and the inhomogeneous term  $F_{n-1}$ . The demonstration repeats that of Section 2. except that now the  $W$  terms substitute only two interaction terms  $V_{1n}$  and  $V_{23}$  present in (35) for  $n$  gluons by two terms  $V_{12,3}$  and  $V_{12,n}$  which should appear for  $n - 1$  gluons.

We now repeat this procedure for  $\psi_{n-1}(q_1 + q_2, q_3, \dots, q_n)$  bootstrapping another pair of neighbours among the  $n - 1$  remaining gluons. Namely we write a particular solution  $\psi_{n-1}$  expressed via an arbitrary solution for  $n - 2$  gluons

$$\psi_{n-1}(q_1, \dots, q_{n-1}) = g \psi_{n-2}(q_1, \dots, q_i + q_{i+1}, \dots, q_{n-1}) \quad (38)$$

Note that  $i$  may be quite arbitrary. In particular we may choose  $i = 1$  or  $i = n - 1$  bootstrapping the gluon which is itself a result of fusion. Continuing this process we finally express the solution to the original equation for  $n$  gluons through the solution  $\psi_2$  for two gluons in a colourless state (the BFKL pomeron):

$$\psi_n(q_1, \dots, q_n) = g^{n-2} \psi_2(\tilde{q}_1, \tilde{q}_2) \quad (39)$$

The momenta of the two gluons  $\tilde{q}_1, \tilde{q}_2$  are sums of momenta of neighbouring gluons whose number and particular choice depend on the bootstrapping process. The function  $\psi_2(\tilde{q}_1, \tilde{q}_2)$  satisfies the standard BFKL equation with a certain inhomogeneous term  $F_2(\tilde{q}_1, \tilde{q}_2)$ . It should

evidently depend on two sums of momenta of  $n$  initial gluons. It is this term which ultimately selects the solution for the whole tower of functions  $\psi_n, \psi_{n-1}, \dots, \psi_2$

For the original equation for  $n$  gluons all contributions which arise as we successively represent  $\psi_{n-1}, \psi_{n-2}, \dots$  appearing in  $F_n, F_{n-1}, \dots$  via  $\psi_2$  represent a specific and rather complicated inhomogeneous term. Thus what we have obtained is only a set of particular solutions of the inhomogeneous Schrödinger equation. We want to argue, however, that this set includes all the physically interesting solutions. Any found solution, as mentioned, can be characterized by the final inhomogeneous term

$$g^{n-2} F_2(\tilde{q}_1, \tilde{q}_2) \quad (40)$$

where  $\tilde{q}_1$  and  $\tilde{q}_2$  are sums of momenta of neighbouring gluons in the original  $n$ -gluon chain. This function represents the contribution to  $\psi_n$  in the lowest order in  $g$ , i.e. the coupling vertex for  $n$  gluons to the target (projectile). Thus it is clear that should this vertex have a more complicated structure (e.g. depend on three or more momenta) then our set of solutions is too small. On the other hand, for any vertex of the form (40) we can consecutively construct  $\psi_2, \psi_3, \dots, \psi_n$  using the bootstrapping procedure for the gluons whose momenta are summed in the arguments of the function (40).

It is difficult to say something definite about the coupling of  $n$  gluons to a general hadronic target because of the nonperturbative effects. However in the case when one can study the coupling perturbatively, namely, for the photon target, the coupling vertex, as will be shown in the next section, indeed depends only on two momenta which are sums of the gluon momenta, i.e. has the form (40). Thus at least in this case the found set of solutions seems to cover the physical ones. As to other contributions appearing in the inhomogeneous term  $F_n$  in the course of bootstrapping, they all take into account processes when first two gluons couple to the target which then consecutively split into more and more gluons until their number becomes  $n$  and they start to interact pairwise with the BFKL kernel. This picture was observed by J.Bartels in the four-gluon system [ 7 ], upon the study of appropriate cuts of the three-particle amplitude in the triple Regge-kinematics.

Thus, if our argument is correct, the  $n$ -gluon chain in the overall colorless state is equivalent to a single BFKL pomeron. Its relative weight is determined by the factor  $(g^2 N_c)^n$  which comes from the coupling to the external source and colour factor. The BFKL pomeron itself corresponds to  $n = 2$  and his weight is  $g^4 N_c^2$ . Evidently all other gluon chains with  $n \geq 3$  give corrections to the BFKL pomeron coupling of the relative order  $(g^2 N_c)^n$ . The parameter  $g^2 N_c$  is however assumed small in this essentially perturbative approach. As a result, in the limit  $N_c \rightarrow \infty, g^2 N_c \rightarrow 0$  we can completely neglect all cylinder configurations and all solutions to the  $n$ -gluon problem reduce to  $n/2$  non-interacting BFKL pomerons, which is nothing but the

normal  $n/2$  cut contribution. Since no interaction remains between pomerons for large  $N_c$ , it is not difficult to sum this leading contribution for all  $n$  and obtain a unitary amplitude. To do that we have to study the coupling of  $n$  gluons to the projectile (target). This will be done in the next section.

## 5 The $n$ -gluon-photon coupling

As mentioned, we choose photons (real or virtual) as the target and projectile, since in this case we can study the coupling to gluons perturbatively. To separate the coupling we consider the  $\gamma^*\gamma$  forward scattering amplitude corresponding to  $n$ -gluon exchange. The procedure we adopt closely follows the well-known one for the two-gluon exchange (see [ 11 ]), so that we shall be brief.

Let the momentum of the virtual photon (projectile) be  $q$  and that of the real photon (target) be  $p$ . We choose a system where  $p_+ = q_-$ ,  $p_- = p_\perp = q_\perp = 0$ . Then  $\nu = pq = p_+q_- = p_+^2 \rightarrow \infty$ ,  $x = -q^2/(2\nu)$  and  $q_+ = -xp_+$ . We shall consider the case of  $x \ll 1$ , so that in what follows we always neglect  $x$  as compared to quantities of the order unity. The forward scattering amplitude corresponding to the  $n$ -gluon exchange is written in the form

$$iA(p, q) = (1/n!) \int \prod_{k=1}^{n-1} (d^4 q_k / (2\pi)^4) i\Gamma_p(q, q_i) i\Gamma_t(p, q_i) \prod_{k=1}^n (i/q_k^2) \quad (41)$$

where  $\sum_{k=1}^n q_k = 0$ . The  $\Gamma$ 's are 4-dimensional vertices for the interaction of the target ( $t$ ) and projectile ( $p$ ) with  $n$  gluons. Vector multiplication in colour and space gluon indeces is understood. Vector indeces should also be associated with the interacting photons. As shown in [ 11 ] at high energies one can factorize the vector summation in space indeces conserving only the longitudinal components of the exchanged gluons, which amounts to changing each metric tensor according to

$$g_{\alpha\beta} \rightarrow p_\alpha q'_\beta / \nu \quad (42)$$

where  $q' = q + xp$  and  $\alpha(\beta)$  refers to the projectile (target). As to the photon vector indeces, we project onto its longitudinal and transversal components by means of the standard projectors

$$P_L^{\alpha\beta} = (q^2/\nu) p^\alpha p^\beta; P_\perp^{\alpha\beta} = (1/2) g_\perp^{\alpha\beta} \quad (43)$$

We are interested in a kinematical situation when the gluons are reggeized. It means that all intermediate states in the two  $\Gamma$ 's should have finite masses (not growing with  $\nu$ ). From that we conclude that  $q_{i\pm}$  are small and  $q_i^2 \simeq q_{i\perp}^2$ . Also one concludes that the projectile vertex  $\Gamma_p$  does not depend on  $q_{i-}$  and the target one  $\Gamma_t$  does not depend on  $q_{i+}$ . This makes it possible to present (41) as a multiple two dimensional integral with a factorized integrand

$$A(p, q) = (4\nu/n!) i^{n+1} \int \prod_{k=1}^{n-1} (d^2 q_k / (2\pi)^2) F_p(q, q_i) F_t(p, q_i) \prod_{k=1}^n (1/q_{k\perp}^2) \quad (44)$$

Applying (42) we find for the projectile part

$$F_p(q, q_i) = (1/2q_-)(q_-/\nu)^n \int \prod_{k=1}^{n-1} (dq_{k+}/(2\pi)) \Gamma_p^{\alpha_1 \dots \alpha_n}(q, q_i) p_{\alpha_1} \dots p_{\alpha_n} \quad (45)$$

where  $q_{i-} = 0$ . This formula defines the 2-dimensional vertex for the interaction of the projectile with  $n$  gluons. For the target we obtain a similar formula with  $p \rightarrow q'$ . It is these vertices that enter the  $n$  gluon equation as inhomogeneous terms.

To calculate the integrals over  $q_{i+}$  in (45) we change to variables

$$s_k = (q + \sum_{i=1}^k q_i)^2, \quad k = 1, \dots, n-1$$

and deform the Feynman integration contour in each  $s_k$  to close around the cut along the positive axis

$$F_p(q, q_i) = (1/2)\nu^{-n} \int \prod_{k=1}^{n-1} (ds_k/4\pi) \text{disc} \Gamma_p(q, s_i) \quad (46)$$

The multiple discontinuity in all variables  $s_1, \dots, s_{n-1}$  should be taken. It can be calculated by using the unitarity condition and inserting some intermediate states. In the lowest order in the coupling constant one retains only the simplest  $q\bar{q}$  state in each discontinuity. The transition from the  $i-1$ th to the  $i$ th intermediate state is accompanied by the emission of the  $i$ th gluon,  $i = 2, \dots, n-1$ , the gluon 1( $n$ ) present in the initial (final) state. Each gluon may be emitted either by the quark or by the antiquark. This gives  $2^n$  different contributions.

This result exactly corresponds to calculating the  $q\bar{q}$  loop with  $n$  gluons attached to it in a specific manner. All  $q$  and  $\bar{q}$  denominators  $(m^2 - k^2)^{-1}$  should be changed to  $2\pi i \delta(m^2 - k^2)$  except the ones which connect the photon vertices with gluons 1 and  $n$ . Also the gluons should follow in the rising order both on the  $q$  and  $\bar{q}$  lines. If the numbers of the gluons emitted from the  $q$  line are  $i_1, \dots, i_{n_1}$  we should have

$$i_1 < i_2 < \dots < i_{n_1} \quad (47)$$

and similarly for the rest  $n - n_1$  gluons attached to the  $\bar{q}$  line. With the order of gluons emitted from  $q$  and  $\bar{q}$  lines fixed, different contributions are obtained only when some of the gluons change their source from  $q$  to  $\bar{q}$  or vice versa. With  $n_1$  gluons emitted from  $q$  and  $n_2 = n - n_1$  gluons emitted from  $\bar{q}$ , the total number of different contributions is  $C_n^{n_1}$ . Summed over  $n_1$  it gives  $2^n$  as it should be.

With all but two gluon propagators changed to  $\delta$ -functions the integrations over the initial quark momentum  $k_{0+}$  and  $q_{i+}$   $i = 1, \dots, n-1$  become trivial. To simplify the presentation we start with the case of an Abelian gauge group, i. e. when there are no colour indeces. Then the only preliminary step left is to take the spinor trace in the loop. The easiest case is the longitudinal photon. Then all vertices in the loop are changed to  $\hat{p} = p_+ \gamma_-$ . Correspondingly

of every propagator numerator only the part  $k_{i-}\gamma_+$  is left. All "−" components of the quark or antiquark momenta are equal because  $q_{i-} \ll k_{i-}$ . The longitudinal trace thus becomes

$$T_L = -(q^2/\nu^2) 2^{n+3} p_+^{n+2} k_{0-}^{n_1+1} (k_{0-} - q_-)^{n_2+1} \quad (48)$$

where  $n_1(n_2)$  is the number of gluons emitted from  $q(\bar{q})$  and  $k_0$  is the quark's momentum before all emissions. For the transverse photon we find two transverse photon vertices  $\gamma_\perp$  which require that in the adjoining propagators the transverse part  $\hat{k}_{i\perp}$  or the mass term should be retained. The transverse trace results

$$T_\perp = -2^{n+1} p_+^n k_{0-}^{n_1-1} (k_{0-} - q_-)^{n_2-1} (m^2 + (k_{0-}^2 + (q_- - k_{0-})^2)(k_0 \tilde{k})_\perp) \quad (49)$$

where  $\tilde{k}$  is the quark's momentum after all emissions.

Let us take the configuration when the gluons  $1, \dots, n_1$  are emitted from the quark and the gluons  $n_1 + 1, \dots, n$  are emitted from the antiquark. Performing the integrations over  $k_{0+}$  and  $q_{i+}$ ,  $i = 1, \dots, n-1$  we find the contribution to the longitudinal vertex  $F_L$

$$\begin{aligned} F_L^{(n_1)}(q_{i\perp}) &= 4i(-1)^{n_1} q^2 e^2 g^n \sum_{f=1}^{N_f} Z_f^2 \int_0^1 d\alpha (\alpha(1-\alpha))^2 \\ &\quad \int (d^2 k / (2\pi)^3) (k_\perp^2 + \epsilon_f^2)^{-1} ((k + \sum_{i=1}^{n_1} q_i)_\perp^2 + \epsilon_f^2)^{-1} \end{aligned} \quad (50)$$

where

$$\epsilon_f^2 = m_f^2 - \alpha(1-\alpha)q^2 \quad (51)$$

The summation goes over quarks of different flavours  $f$  with masses  $m_f$  and charges  $Z_f e$ ;  $\alpha$  is the scaling variable for the quark:  $k_{0-} = \alpha q_-$ . The transverse vertex has a similar structure

$$\begin{aligned} F_\perp^{(n_1)}(q_{i\perp}) &= i(-1)^{n_1} e^2 g^n \sum_{f=1}^{N_f} Z_f^2 \int_0^1 d\alpha \int (d^2 k / (2\pi)^3) \\ &\quad (m_f^2 + (\alpha^2 + (1-\alpha)^2)(k, k + \sum_{i=1}^{n_1} q_i)_\perp^2) (k_\perp^2 + \epsilon_f^2)^{-1} ((k + \sum_{i=1}^{n_1} q_i)_\perp^2 + \epsilon_f^2)^{-1} \end{aligned} \quad (52)$$

Both contributions depend only on the sum  $\sum_{i=1}^{n_1} q_i$  of the gluon momenta emitted from the quark (or antiquark). The total vertices are obtained upon summing over all different distributions of gluons between the quark and antiquark.

Now we introduce colour variables. We first consider the case when gluons form colourless pairs, so that their colour wave function has the structure (19). In this case the colour matrices of the quark loop  $t^a$ ,  $a = 1, \dots, N_c$  should be summed pairwise over  $a$ . Diagrammatically it is equivalent to joining the corresponding loop vertices by a gluon line (only in colour space). In the limit  $N_c \rightarrow \infty$  the leading contribution will come from configurations in which all these

$n/2$  gluon colour lines do not cross, i.e the loop with these spurious colour lines is planar. In such a planar loop all different pairs  $t^a t^a$  can be subsequently put together and summed over  $a$  to give  $(N_c^2 - 1)/N_c \sim N_c/2$  each. The colour trace then becomes

$$c_n = N_c(N_c/2)^{n/2} \quad (53)$$

Turning to our derivation of the vertex without colours, we observe that with the order of gluons fixed by the condition (47), the gluon colour lines joining the pairing loop vertices never cross. In fact, suppose that two lines connecting quark vertices  $i, i+1$  and  $k, k+1$ ,  $i < k$ , cross. This may only happen if  $k+1 < i+1$ , which is impossible. Similarly one shows that gluon lines do not cross if they start on the quark line and finish on the antiquark line. As a result, introduction of colour for the colourless pair configuration does not change the vertices found for the Abelian case except for the overall colour factor  $c_n$ , Eq. (53).

A different result is obtained for the configuration in which gluons are in the adjoint representation with their neighbours and form a cylinder with the two  $q\bar{q}$  loops as bases. Each gluon together with the quark from the target and antiquark from the projectile (or vice versa) then contribute a factor  $N_c$ , giving an overall colour factor  $N_c^n$  (for the amplitude). Comparing with the Abelian case we find that we have to retain only a part of the configurations which correspond to a fixed order of the gluons along the  $q\bar{q}$  loop (say,  $1, \dots, n$ ) arbitrarily divided between the quark and antiquark. As a result only  $n$  terms survive for each  $n_1 \neq 0, n$  and not  $C_n^{n_1}$  as in the previous cases. Still the obtained vertex always depends on only two momenta  $\tilde{q}_1$  and  $\tilde{q}_2$  which are sums of the momenta of neighbouring gluons (in fact, for the forward scattering  $\sum_{i=1}^n q_i = 0$  so that  $\tilde{q}_1 + \tilde{q}_2 = 0$  and the vertex actually depends on only one momentum). This fact was used in the previous section to argue that the solution obtained for the chain Hamiltonian was general enough.

## 6 The photon structure function at low $x$

With the leading contribution given by  $n/2$  non-interacting pomerons and the vertices for the  $n$ -gluon coupling to the target and projectile known, we can proceed to sum contributions from all  $n$  to obtain a unitary description for the  $\gamma^*\gamma$  scattering. We assume that gluons are paired into BFKL pomerons in the order  $(12), (34), \dots, (n-1, n)$ . In fact we have  $(n-1)!!$  different orderings, which all give the same contribution. This changes the symmetry factor according to

$$(n-1)!!/n! = 1/(2^{n/2}(n/2)!) \quad (54)$$

The factor  $(1/2)^{n/2}$  corresponds to the symmetry factor  $1/2$  for each pair;  $1/(n/2)!$  is the correct symmetry factor for  $n/2$  pomerons.

To perform the summation over  $n$  we first present the sum of all contributions (51) or (52) to the vertices coming from different distributions of the gluons between  $q$  and  $\bar{q}$  in a convenient manner. We use the approach proposed in [ 12 ] based on the impact parameter space. One writes

$$(k_\perp^2 + \epsilon_f^2)^{-1} = \int (d^2 r / 2\pi) K_0(\epsilon_f r)$$

where  $K_0$  is the McDonald function. Then the contribution (51) from the given gluon distribution to the longitudinal vertex is rewritten as (with the colour factor (53))

$$\begin{aligned} F_L^{(n_1)}(q_{i\perp}) &= 4ic_n(-1)^{n_1} q^2 e^2 g^n \\ &\sum_{f=1}^{N_f} Z_f^2 \int_0^1 d\alpha (\alpha(1-\alpha))^2 \int (d^2 r / (2\pi)^3) K_0^2(\epsilon_f r) \exp(ir \sum_{i=1}^{n_1} q_i) \end{aligned} \quad (55)$$

Following [ 12 ] we introduce the longitudinal  $q\bar{q}$  density of the projectile in the  $\alpha, r$  space

$$\rho_L(\alpha, r) = \sum_{f=1}^{N_f} Z_f^2 (\alpha(1-\alpha))^2 K_0^2(\epsilon_f r) \quad (56)$$

The sum over all distributions results as

$$\begin{aligned} F_L^{(n_1)}(q_{i\perp}) &= 4ic_n(-1)^{n_1} q^2 e^2 g^n \\ &\int_0^1 d\alpha \int (d^2 r / (2\pi)^3) \rho_L(\alpha, r) \prod_{i=1}^n (\exp(ir q_i) - 1) \end{aligned} \quad (57)$$

In the same way for the transverse vertex we introduce the transverse density of the projectile

$$\rho_\perp(\alpha, r) = \sum_{f=1}^{N_f} Z_f^2 (m_f^2 K_0^2(\epsilon_f r) + (\alpha^2 + (1-\alpha)^2) \epsilon_f^2 K_1^2(\epsilon_f r)) \quad (58)$$

The transverse vertex aquires the form

$$\begin{aligned} F_\perp^{(n_1)}(q_{i\perp}) &= ic_n(-1)^{n_1} e^2 g^n \\ &\int_0^1 d\alpha \int (d^2 r / (2\pi)^3) \rho_\perp(\alpha, r) \prod_{i=1}^n (\exp(ir q_i) - 1) \end{aligned} \quad (59)$$

Analogous expressions are obtained for the target (real and transverse) photon with  $q^2 = 0$ .

Now we can calculate the contribution to the amplitude coming from the exchange of  $n/2$  non-interacting pomerons. We have only to change pairs of gluon propagators in (44) by BFKL pomerons taken in the energy representation. Let  $l = q_1 + q_2 = q'_1 + q'_2$  be the total momentum of the two gluons 1 and 2 forming a pomeron,  $k = (1/2)(q_1 - q_2)$  and  $k' = (1/2)(q'_1 - q'_2)$  be their final and initial relative momenta. Then we have to change

$$(2\pi)^2 \delta^3(k - k') / q_1^2 q_2^2 \rightarrow f(\nu, l, k, k') \quad (60)$$

where

$$f(\nu, l, k, k') = \int (dE/2\pi i) \nu^{-E} \psi_2(E, l, k, k') / q_1^2 q_2^2 \quad (61)$$

and  $\psi_2(E, l, k, k')$  is the solution of Eq. (1) for two gluons in a colourless state with the inhomogeneous term  $F_2 = (2\pi)^2 \delta^2(k - k')$ . The integration in (61) runs along the imaginary axis to the left of all singularities of the integrand in  $E$ . We find for the contribution from the  $n/2$  pomeron exchange to the  $\gamma^* \gamma$  forward scattering amplitude (for the transverse projectile photon)

$$\begin{aligned} A_{\perp}^{(n)}(p, q) &= -4i\nu e^4 N_c^2 (-g^4/16)^{n/2} (1/(n/2)!) \int_0^1 d\alpha \int_0^1 d\beta \\ &\int d^2 R \int d^2 r \int d^2 r' (2\pi)^{-6} \int \prod_{i=1}^{n/2} (d^2 l d^2 k d^2 k' (2\pi)^{-6} f(\nu, l_i, k_i, k'_i) \exp(il_i R)) \\ &\rho_{\perp}(\alpha, r) \rho_{\perp}(\beta, r', \epsilon_f = m_f) \prod_{i=1}^n (\exp(ir q_i) - 1) (\exp(-ir' q'_i) - 1) \end{aligned} \quad (62)$$

The  $R$  integration takes into account the condition  $\sum_{i=1}^{n/2} l_i = 0$

The integration over the pomeron momenta gives

$$\begin{aligned} &\int (d^2 l d^2 k d^2 k' / (2\pi)^6) f(\nu, l, k, k') \exp(ilR) (\exp iq_1 r - 1) (\exp iq_2 r - 1) \\ &(\exp(-iq'_1 r') - 1) (\exp(-iq'_2 r') - 1) = 4(2\pi)^{-4} f(\nu, R + (1/2)(r - r'), r, r') \end{aligned} \quad (63)$$

where  $f(\nu, R, r, r')$  is the BFKL correlator in the impact parameter space found in [ 13 ]. Retaining only the  $s$ -wave contribution dominant at large  $\nu$

$$f(\nu, R, r, r') = \int d\kappa \frac{\kappa^2 \nu^{-E(\kappa)}}{(\kappa^2 + 1/4)^2} \int d^2 r_0 \left( \frac{r}{r_{10} r_{20}} \right)^{1+2i\kappa} \left( \frac{r'}{r'_{10} r'_{20}} \right)^{1-2i\kappa} \quad (64)$$

where

$$r_{10} = R + r/2 - r_0, \quad r_{20} = R - r/2 - r_0, \quad r'_{10} = r'/2 - r_0, \quad r'_{20} = -r'/2 - r_0$$

and  $E(\kappa)$  is the BFKL pomeron energy

$$E(\kappa) = (g^2 N_c / 2\pi^2) (\text{Re } \psi(1/2 + i\kappa) - \psi(1)) \quad (65)$$

Under the sign of integrals over  $\alpha, \beta, R, r$  and  $r'$  in (62) we find the  $n/2$ th power of  $f(\nu, R, r, r')$ . Evidently the contribution has an eikonal form. Summing over  $n$  we finally obtain the total amplitude for the transverse projectile as an integral of the known functions

$$\begin{aligned} A_{\perp}(p, q) &= 4i\nu e^4 N_c^2 \int_0^1 d\alpha \int_0^1 d\beta \\ &\int d^2 R \int d^2 r \int d^2 r' (2\pi)^{-6} \rho_{\perp}(\alpha, r) \rho_{\perp}(\beta, r', q^2 = 0) (1 - \exp(-\frac{g^4}{4(2\pi)^4} f(\nu, R, r, r'))) \end{aligned} \quad (66)$$

For the longitudinal projectile photon one has to change  $\rho_{\perp}(\alpha, r)$  to  $\rho_L(\alpha, r)$ .

The amplitude (66) represents the leading contribution to the  $\gamma^*\gamma$  forward scattering amplitude in the limit  $N_c \rightarrow \infty$ . It has an eikonal structure and is evidently unitary.

At large  $\nu$  small values of  $\kappa$  dominate in (64) where

$$E(\kappa) = E_0 + a\kappa^2, \quad a = (7/2)\zeta(3) \quad (67)$$

and  $E_0$  is negative (see (33)). This gives the well-known large factor  $\nu^{|E_0|}$  in  $f$ . From the structure of  $f$  it is then clear that the dominant contribution comes from the region of small  $r$  and  $r'$ . In this region  $f(\nu, R, r, r')$  results a function of only two scalar variables  $\nu$  and  $z = R^2/rr' \gg 1$ :

$$f(\nu, R, r, r')_{r,r' \ll R} \simeq \frac{16\pi^{3/2}\nu^{|E_0|}}{(a \ln \nu)^{3/2}} z^{-1} \ln z \exp(-\frac{\ln^2 z}{a \ln \nu}) \quad (68)$$

where  $a$  is given in (67).

Passing in (66) to the integration over  $z$  we find that in the asymptotical region  $\nu \rightarrow \infty$  the amplitude completely factorizes

$$A_\perp(p, q) = G_\perp(q^2)F(\nu)G_\perp(p^2) \quad (69)$$

Here  $F(\nu)$  is the contribution from all exchanged pomerons (the "Froissaron") and  $G_\perp(q^2)$  and  $G_\perp(p^2)$  are its effective couplings to the projectile and target respectively. For the projectile

$$G_\perp(q^2) = (e^2 N_c / 4\pi) \int_0^1 d\alpha \int_0^\infty r^2 dr \rho_\perp(\alpha, r) \quad (70)$$

In the limit  $Q^2 = -q^2 \rightarrow \infty$  we find

$$G_\perp(q^2) = \frac{e^2 N_c a_1}{2Q} \sum_f Z_f^2 \quad (71)$$

where  $a_n$  are numbers defined by

$$a_n = \int_0^\infty r^2 dr K_n^2(r) \quad (72)$$

For the real photon as a target ( $p^2 = 0$ )

$$G(0)_\perp = \frac{e^2 N_c (a_0 + 2a_1)}{4\pi} \sum_f Z_f^2 / m_f \quad (73)$$

From (71) we observe that the cross-section falls only as  $1/Q$  at large  $Q^2$  instead of the standard scaling behaviour  $1/Q^2$ . This means that the structure function rises roughly as  $Q$ .

The  $\nu$  dependence is determined by the Froissaron  $F(\nu)$ :

$$F(\nu) = (i\nu/\pi) \int_1^\infty dz (1 - \exp(-bz^{-1} \ln z \exp(-\xi \ln^2 z))) \quad (74)$$

where according to (68)

$$\xi = (a \ln \nu)^{-1}, \quad b = g^4 \nu^{|E_0|} \xi^{3/2} / (4\pi^{3/2})$$

To crudely estimate the behaviour of  $F(\nu)$  at large  $\nu$  we approximately determine the point  $x_0 \gg 1$  where the exponent becomes small:  $x_0 \sim \nu^\lambda$  where

$$\lambda = (1/2)(\sqrt{a^2 + 4a|E_0|} - a) \quad (75)$$

Then neglecting the exponential function in the region  $x < x_0$  we obtain

$$F(\nu) \simeq (i/\pi) \nu^{1+\lambda} \quad (76)$$

Thus after summing over all pomeron exchanges the cross-section remains growing as a power  $\lambda$  of the c.m. energy squared (somewhat smaller than the original power  $|E_0|$ ). Correspondingly the small  $x$  behaviour of the structure function is a power growth

$$F_2(Q^2, x) \sim Q^{1+2\lambda} x^{-\lambda} \quad (77)$$

The obtained undesirable features of the seemingly unitary  $\gamma^* \gamma$  amplitude are evidently related to the scaling invariant character of the theory. With the gluon mass equal to zero and no confinement the projectile and target can interact at arbitrary large distances. We hope that introduction of the running coupling and the QCD scale  $\Lambda$  will lead to a more realistic picture.

## 7 Conclusions

We have generalized the bootstrap relation to the case of  $n$  reggeized gluons. It allows to conclude that the intercept generated by  $n$  gluons cannot be lower than for  $n = 2$ . In particular the odderon intercept ( $n = 3$ ) cannot be lower than the pomeron one, so that the variational estimate obtained in [ 14 ] is too low. With a  $q\bar{q}$  loop as the external source, the odderon possesses the same intercept as the BFKL pomeron.

The most far-reaching consequences seem to follow from the bootstrap in the limit  $N_c \rightarrow \infty$  for the gluon chain with only neighbours interacting. We have given arguments that all physically interesting solutions of this system reduce to a single BFKL pomeron. This conclusion means that the BFKL pomeron coincides with the topological pomeron, which corresponds to a sum of all diagrams with a topology of a cylinder in the large  $N_c$  limit [ 15 ].

The leading contribution from the  $n$ -gluon system in the large  $N_c$  limit comes from  $n/2$  non-interacting BFKL pomerons. We have summed this contribution to the  $\gamma^* \gamma$  forward

scattering amplitude for all  $n$  and obtained a closed formula of an eikonal form expressed through the known functions. Crude estimates reveal that the resulting cross-section continues to grow as a power with energy and falls as  $1/Q$  with the virtuality of the photon projectile  $q^2 = -Q^2$ , which leads to a gross violation of scaling.

These results were obtained in the standard framework of the BFKL theory, namely, for an infrared regularized theory with a small fixed coupling (and hence no confinement). One should be most cautious in trying to extend them to the realistic QCD. For such an extension the running coupling constant and the corresponding QCD scale have to be introduced.

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